A bi-directional extensible ad hoc interface between Lean and Mathematica

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Computer algebra systems

Strengths:

- They are easy and useful
- They provide instant gratification
- They support interactive use, exploration
- They are programmable and extensible

Weaknesses:

- The focus is on symbolic computation, rather than abstract definitions and assertions
- They are not designed for reasoning or search
- The semantics is murky
- They are sometimes inconsistent

Interactive theorem provers

Strengths:

- Their languages are expressive and well-specified
- They come with a precise semantics
- Results are fully verified

Weaknesses:

- Formalization is slow and tedious
- It requires a high degree of commitment and expertise
- It doesn't promote exploration and discovery

ITP + CAS

By linking the two, we can

- Allow exploration and computation in the proof assistant, without reimplementing algorithms
- Lower the barrier for newcomers to ITP
- Loan a semantics/proof language to CAS

Many projects have attempted to connect the two: verified CAS algorithms, trusting links, verified links, ephemeral links, CAS proof languages.

Contributions

- An extensible procedure to interpret Lean in Mathematica
- An extensible procedure to interpret Mathematica in Lean
- A link allowing Lean to evaluate arbitrary Mathematica commands, and receive the results
- Tactics for certifying results of particular Mathematica computations
- A link allowing Mathematica to execute Lean tactics and receive the results

Broader picture

- Proof assistants, with clear semantics, can serve as glue between many different mathematical tools.
- Formal mathematical corpora can (should?) contain hooks to external tools.

Outline

Background: Lean

Lean is a new interactive theorem prover, developed principally by Leonardo de Moura at Microsoft Research, Redmond.

Calculus of constructions with:

- Non-cumulative hierarchy of universes
- Impredicative `Prop`
- Quotient types and propositional extensionality
- Axiom of choice available

See `http://leanprover.github.io`, or Mario's talk.

Metaprogramming in Lean

Question: How can one go about writing tactics and automation?

Lean's answer: go meta, and use Lean itself.

Ebner, Ullrich, Roesch, Avigad, de Moura. *A Metaprogramming Framework for Formal Verification*, ICFP 2017.

Metaprogramming in Lean

Advantages:

- Users don't have to learn a new programming language.
- The entire library is available.
- Users can use the same infrastructure (debugger, profiler, etc.).
- Users develop metaprograms in the same interactive environment.
- Theories and supporting automation can be developed side-by-side.

Metaprogramming in Lean

The strategy: expose internal data structures as `meta` declarations, and insert these internal structures during evaluation.

```
meta constant expr : Type
meta constant environment : Type
meta constant tactic_state : Type
meta constant to_expr : expr → tactic expr
meta constant run_io {α : Type} : io α → tactic α
```

Background: Mathematica

Mathematica is a powerful and popular computer algebra system developed at Wolfram Research, implementing the Wolfram Language.

It provides a vast variety of functions for manipulating mathematical expressions, as well as tools for manipulating and displaying data.

Background: Mathematica

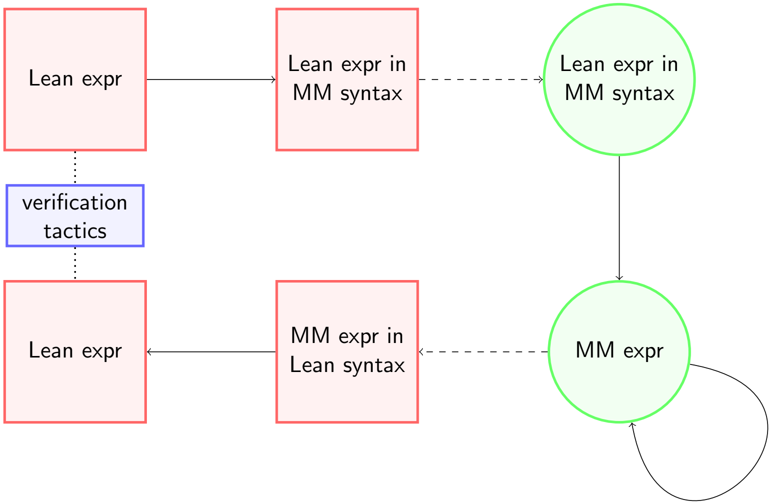
Some basic Mathematica syntax and terminology:

- Function application
  - `Plus[x, y]`
  - `Plus[x, y, z]`
  - `x + y + z`
  - `Factor[x^2 - 2x + 1]`
  - `x^2 - 2x + 1 // Factor`
- In `Plus[x, y]`, we refer to `Plus` as the *head symbol* and `x`, `y` as the *arguments*.
- Head symbols can be given computational behavior via pattern matching rules: `MyFunc[s_String] := Reverse[s]`.

Outline

Link architecture

We'll focus on using Mathematica from within Lean.



Applications

- Factoring (integers, polynomials, matrices)
- Finding solutions (polynomial systems, trig systems)
- Linear arithmetic
- Computing integrals
- Finding counterexamples
- Oracular simplifier/evaluator

Lean expression grammar

```
meta inductive expr (elaborated : bool := tt)
| var         {} : nat → expr
| sort        {} : level → expr
| const       {} : name → list level → expr
| mvar           : name → expr → expr
| local_const    : name → name → binder_info → expr → expr
| app            : expr → expr → expr
| lam            : name → binder_info → expr → expr → expr
| pi             : name → binder_info → expr → expr → expr
| elet           : name → expr → expr → expr → expr
| macro          : macro_def → list expr → expr

meta def mathematica_form_of_level : level → string := ...
meta def mathematica_form_of_name : name → string := ...
meta def mathematica_form_of_expr : expr → string := ...
```

Lean expression grammar

```
x : real ⊢ sin x : real

app (const `sin [])
   (local_const `x `x binder_info.default (const `real []))
```

Applying `mathematica_form_of_expr` produces:

```
App[Const[``sin'', LListNil],
    LocalConst[``x'', ``x'', BID,
                Const[``real'', LListNil]]]
```

Mathematica interpretation rules

The head symbols `App`, `Const`, etc. are uninterpreted in Mathematica.

We want to exploit the facts that:

- certain Lean constants correspond to certain Mathematica constants.
  - **`sin`** "means" the same as `Sin`
- certain expression patterns in Lean correspond to certain expression patterns in Mathematica.
  - λ**`x, t`** "means" the same as `Function[x, t]`

Mathematica interpretation rules

We define a Mathematica function `LeanForm` using pattern matching rules:

```
LeanForm[App[App[App[App[
    Const["add", _], _], _], x_], y_]] :=
  Inactive[Plus][LeanForm[x],LeanForm[y]]

LeanForm[App[Const["list.nil", _], _]] :=
  {}

LeanForm[App[App[App[
    Const["list.cons", _], _], h_], t_]] :=
  Join[{LeanForm[h]}, LeanForm[t]]
```

Mathematica interpreptation rules

With the right set of `LeanForm` rules, we can reduce a Lean expression to something semantically meaningful in Mathematica.

⟦2+3⟧ `// LeanForm` evaluates to `Inactive[Plus][2, 3]`

In a local context with `x : real`,
⟦`x^2 - 2*x + 1`⟧ `// LeanForm // Activate // Factor`
evaluates to `Power[Plus[-1,`⟦`x`⟧`], 2]`

Mathematica expression grammar

```
inductive mmexpr
| sym  : string → mmexpr
| str  : string → mmexpr
| int  : int → mmexpr
| real : float → mmexpr
| app  : mmexpr → list mmexpr → mmexpr
```

Mathematica expressions are built out of atoms and applications of expressions to lists of expressions.

Analogous to the representation of Lean expressions in Mathematica, we can represent any Mathematica expression in Lean with a term of type `mmexpr`.

It is easy to implement
`parse_to_mmexpr : string → tactic mmexpr`.

Lean interpretation rules

We define `pexpr_of_mmexpr : mmexpr → tactic pexpr`.

New interpretation rules can be declared in Lean:

```
@[translation_rule]
meta def list_to_pexpr : app_to_pexpr_keyed_rule :=
⟨"List",
 λ ctx args,
  do args' ← args.mfor (pexpr_of_mmexpr ctx),
     return $ args'.foldr (λ h t, ``(%%h :: %%t)) ``([])⟩
```

translates

`List[x, y, z]` to `cons x (cons y (cons z nil))`.

Combining components

We communicate with a Mathematica kernel using Lean's IO monad.

A tactic to factor a polynomial expression:

```
meta def factor (e : expr) : tactic expr :=
do tp ← infer_type e,
   pe ← evaluate_command_on_expr
          (λ s, s ++ "// LeanForm // Activate // Factor")
          e,
   to_expr ``(%%pe : %%tp)
```

Outline

Certification

There's little reason to trust the output of Mathematica, and less reason to trust this translation process.

For many computations, verifying that a result has some property is easier than computing the result itself.
E.g. if `p` is a polynomial, it is easier to verify that `Factor[p] = p` than to compute `Factor[p]`.

We pair this translation procedure with a set of task-specific verification procedures.

Example: factoring polynomials

```
meta def eq_by_simp (e1 e2 : expr) : tactic expr :=
{ do gl ← mk_app ‘eq [e1, e2],
     mk_inhabitant_using gl ‘[simp]}
<|> fail "unable to simplify"

meta def assert_factor (e : expr) (nm : name) : tactic unit :=
do fe ← factor e,
   pf ← eq_by_simp e fe,
   note nm pf

example (x : ℝ) : 1 - 2*x + 3*x^2 - 2*x^3 + x^4 ≥ 0 :=
begin
 assert_factor 1 - 2*x + 3*x^2 - 2*x^3 + x^4 using h,
 rewrite h,
 apply sq_nonneg
end
```

Example: linear arithmetic

Motzkin transposition theorem

Let P, Q, R be matrices and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ vectors.

- $P\mathbf{x} > \mathbf{p}, Q\mathbf{x} \geq \mathbf{q}, R\mathbf{x} = \mathbf{r}$ has no solution $\mathbf{x}$

if and only if

- $P^T\mathbf{y}_1 + Q^T\mathbf{y}_2 + R^t\mathbf{y}_3 = 0$ has a solution $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ with $\mathbf{y}_1, \mathbf{y}_2 \geq 0$ and either
  - $\mathbf{y}_1 \cdot \mathbf{p} + \mathbf{y}_2 \cdot \mathbf{q} + \mathbf{y}_3 \cdot \mathbf{r} < 0$ or
  - $\mathbf{y}_1 \cdot \mathbf{p} + \mathbf{y}_2 \cdot \mathbf{q} + \mathbf{y}_3 \cdot \mathbf{r} \leq 0$ and $\mathbf{y}_1 > 0$.

This theorem gives a notion of certificate for linear arithmetic.

Example: linear arithmetic

We can use Mathematica to generate witnesses to the MTT and apply them to produce fully checkable proofs.

```
example (a b c d e f g : ℤ)
 (h1 : 1*a + 2*b + 3*c + 4*d + 5*e + 6*f + 7*g ≤ 30)
 (h2 : (-1)*a ≤ 4)
 (h3 : (-1)*b + (-2)*d ≤ -4)
 (h4 : (-1)*c + (-2)*f ≤ -5)
 (h5 : (-1)*e ≤ -3)
 (h6 : (-1)*g ≤ -2) : false :=
by not_exists_of_linear_hyps h1 h2 h3 h4 h5 h6
```

Example: sanity checking

Many computations in Mathematica are not easily certifiable, but can still be useful in interactive proofs.

`sanity_check` runs the Mathematica command `FindInstance` to search for an assignment satisfying the hypotheses and the negation of the goal. The tactic fails if an assignment is found.

```
example (x : ℝ) (h1 : sin x = 0) (h2 : cos x > 0) :
        x = 0 :=
by sanity_check; admit

example (x : ℝ) (h1 : sin x = 0) (h2 : cos x > 0)
        (h3 : -pi < x ∧ x < pi) : x = 0 :=
by sanity_check; admit
```

Example: Mathematica as an oracle

There is a spectrum of trust levels. Some users may be comfortable using Mathematica as an oracle.

```
meta def full_simp (e : expr) : tactic (expr × expr) :=
do pe ← evaluate_command_on_expr
        (λ t, t ++ "//LeanForm//Activate//FullSimplify")
        e,
   eqtp ← to_expr ``(%%e = %%pe),
   ax_name ← add_axiom eqtp,
   proof ← mk_const ax_name,
   return (val, proof)

example (x : ℝ) :
       x*BesselJ 2 x + x*BesselJ 0 x = 2*BesselJ 1 x :=
by prove_by_full_simp
```

Example: Mathematica as an oracle

We can also use Mathematica to obtain approximations of constants and axiomatize these bounds:

```
example :=
begin
  approx (100*BesselJ 2 0.52) (0.00001 : ℝ),
  trace_state
end

/-
approx : 12887461 / 3900000 < 100 * BesselJ 2 (13 / 25)
        ∧ 100 * BesselJ 2 (13 / 25) < 12887539 / 3900000
⊢ true
-/
```

Outline

Calling Lean from Mathematica

Mathematica has no built in notion of “proof” or “correctness.” For some functions, it’s not even clear what the intended semantics are.

We can consider a “proposition” to be true if applying `FullSimplify` evaluates to `True`, but this is of limited scope, and `FullSimplify` is a black box.

Idea: translate Mathematica “propositions” to Lean, where they have semantics and a proof language.

Calling Lean from Mathematica

`ProveUsingLeanTactic[p_, tac_]` takes

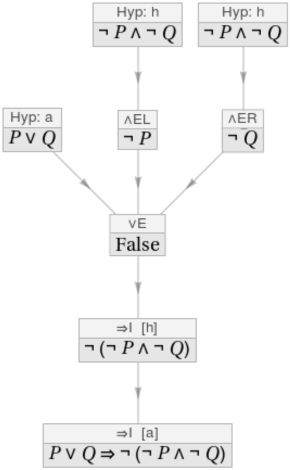
- an expression `p`
- a Lean tactic string `tac`

It translates `p` into a Lean expression `p'`, attempts to prove `p'` using `tac`, and returns the resulting proof term.

Calling Lean from Mathematica

We can add rules to (try to) translate the resulting proof.

```
DiagramOfFormula[
 ForAll[{P, Q},
  Implies[
   Or[P, Q],
   Not[And[Not[P], Not[Q]]]
  ]
 ]
]
```



Calling Lean from Mathematica

We can try to:

- Verify the output of `FullSimplify` or other computations
- Discover missing side conditions
- Extract “interesting” parts of proofs
- “Type-check” or “elaborate” certain Mathematica expressions
- Explore the Lean library

Thanks for listening!

FYI:

Lean Together 2019
January 7-11
Amsterdam, The Netherlands
https://lean-forward.github.io/lean-together