Formalizing a sophisticated definition

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joint work with

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Extending functions

Theorem

Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f \colon A \to \mathbb{R}^q$ extends to a (uniformly) continuous function $\bar{f} \colon \mathbb{R}^p \to \mathbb{R}^q$.



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For every $x\in\mathbb{R}^p$, choose a sequence $a:\mathbb{N}\to A$ converging to x. Uniform continuity of f ensures $f\circ a$ is Cauchy, completeness of \mathbb{R}^q gives a limit y. Set $\bar{f}(x)=y.$ Then prove continuity of $\bar{f}.$

 $\mathsf{Example} \colon \, (+) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \subset \mathbb{R} \text{ extends to } (+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$



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Example: $(+): \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \subset \mathbb{R}$ extends to $(+): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

But multiplication or inversion are *not* uniformly continuous.



Theorem

 $A\subset\mathbb{R}^p$ dense subset. If $f:A o\mathbb{R}^q$ is continuous and

$$\forall x \in \mathbb{R}^p, \exists y \in \mathbb{R}^q, \forall u : \mathbb{N} \to A, u_n \longrightarrow x \Rightarrow f(u_n) \longrightarrow y$$

then f extends to a continuous function $\bar{f} \colon \mathbb{R}^p \to \mathbb{R}^q$.

This applies to multiplication $\mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$.



A better framework?

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We can still say that f(x) converges to y when x tends to x_0 while remaining in A:

$$\forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \forall a \in A \cap V, f(a) \in W.$$



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Theorem

Let X be a topological space, A a dense subset of X, and $f:A\to Y$ a continuous mapping of A into a regular space Y. If, for each $x_0\in X$, f(x) tends to a limit in Y when x tends to x_0 while remaining in A then f extends to a continuous map $\bar{f}:X\to Y$



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Better framework:

$$\begin{array}{c}
A \xrightarrow{f} Y \\
\downarrow \downarrow & \exists ?\bar{f} \\
X
\end{array}$$

Issue: will we need discussions of



Side issue: how to formally refer to f?

extend f i de h where de is a proof that i is a dense topological embedding, and h is a proof that f admits a limit...?

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A better solution is to define an extension operator E_i by:

$$E_i(f)(x) = \begin{cases} \text{some } y \text{ such that } f(a) \text{ tends to } y \text{ when } a \text{ tends to } x \\ \text{some junk value if no such } y \text{ exists} \end{cases}$$

Density of image of i is used only to ensure Y is non-empty! Then use de.extend ${\bf f}$



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The i map is *not* injective if $\{0\}$ is not closed in R.



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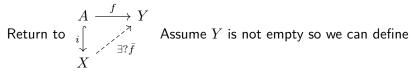
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Note: Even in ZFC, if R is already separated, $R' \neq R$.



Final extension theorem



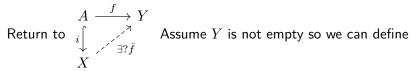
 E_i without any assumption on i.

Theorem

Fix $x_0 \in X$. If, for every x_1 in a neighborhood of x_0 , f(x) tends to a limit in Y when x tends to x_1 while remaining in A then $E_i f$ is continuous at x_0 .



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 E_i without any assumption on i.

Theorem

Fix $x_0 \in X$. If, for every x_1 in a neighborhood of x_0 , f(x) tends to a limit in Y when x tends to x_1 while remaining in A then E_if is continuous at x_0 .

If in addition $x_0=i(a)$, f is continuous at a and i pulls back the topology of X to the topology of A then $E_i(f)(i(a))=f(a)$.



Side propaganda

The assumption in the first part of the previous theorem can be written as

$$\begin{split} \exists U \in \mathcal{N}_{x_0}, \forall x \in U, \exists y \in Y, \forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \\ \forall a \in A, i(a) \in V \Rightarrow f(a) \in W. \end{split}$$



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In mathlib, the assumption is written as

$$\{x\mid \exists y, f_*i^*\mathcal{N}_x\leq \mathcal{N}_y\}\in \mathcal{N}_{x_0}.$$



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- Big projects are good. Next one?

