Formalizing a sophisticated definition

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joint work with

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Formal Methods in Mathematics – Lean Together
January 7th 2020

Extending functions

Theorem

Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f\colon A \to \mathbb{R}^q$ extends to a (uniformly) continuous function $\bar{f}\colon \mathbb{R}^p \to \mathbb{R}^q$.

Extending functions

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For every $x \in \mathbb{R}^p$, choose a sequence $a : \mathbb{N} \to A$ converging to x. Uniform continuity of f ensures $f \circ a$ is Cauchy, completeness of $\mathbb{R}^q$ gives a limit y. Set $\bar{f}(x) = y$. Then prove continuity of $\bar{f}$.

Example: $(+) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \subset \mathbb{R}$ extends to $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

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But multiplication or inversion are *not* uniformly continuous.

Theorem

$A \subset \mathbb{R}^p$ *dense subset. If* $f : A \to \mathbb{R}^q$ *is continuous and*

$$\forall x \in \mathbb{R}^p, \exists y \in \mathbb{R}^q, \forall u : \mathbb{N} \to A, u_n \longrightarrow x \Rightarrow f(u_n) \longrightarrow y$$

then f *extends to a continuous function* $\bar{f} \colon \mathbb{R}^p \to \mathbb{R}^q$.

This applies to multiplication $\mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$.

A better framework?

In order to handle inversion $\mathbb{Q}^* \to \mathbb{R}^*$ and more general spaces, we want a version where $\mathbb{R}^p$ and $\mathbb{R}^q$ are replaced by general topological spaces X and Y.

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We can still say that $f(x)$ converges to y when x tends to x_0 while remaining in A:

$$\forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \forall a \in A \cap V, f(a) \in W.$$

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Theorem

Let X be a topological space, A a dense subset of X, and $f : A \to Y$ a continuous mapping of A into a regular space Y. If, for each $x_0 \in X$, $f(x)$ tends to a limit in Y when x tends to x_0 while remaining in A then f extends to a continuous map $\bar{f} : X \to Y$

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Better framework:

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\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\
X \arrow[ur, dashed, "\exists ?\, \bar{f}"'] &
\end{tikzcd}
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Issue: will we need discussions of

$$\begin{tikzcd}
\mathbb{Q} \arrow[d, hook] & \mathbb{Q}^* \arrow[l, hook'] \arrow[r, "inv"] \arrow[d, hook] & \mathbb{Q}^* \arrow[r, hook] \arrow[d, hook] & \mathbb{Q} \arrow[d, hook] \\
\mathbb{R} & \mathbb{R}^* \arrow[l, hook'] \arrow[r, "inv"] & \mathbb{R}^* \arrow[r, hook] & \mathbb{R}
\end{tikzcd}$$

Side issue: how to formally refer to $\bar{f}$?

`extend f i de h` where `de` is a proof that i is a dense topological embedding, and `h` is a proof that f admits a limit...?

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A better solution is to define an extension operator E_i by:

$$E_i(f)(x) = \begin{cases} \text{some } y \text{ such that } f(a) \text{ tends to } y \text{ when } a \text{ tends to } x \\ \text{some junk value if no such } y \text{ exists} \end{cases}$$

Density of image of i is used only to ensure Y is non-empty!
Then use `de.extend f`

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The i map is *not* injective if $\{0\}$ is not closed in R.

Standard solution

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Note: Even in ZFC, if R is already separated, $R' \neq R$.

Final extension theorem

Return to
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ X \arrow[ur, dashed, "\exists ? \bar{f}"'] & \end{tikzcd}$$
Assume Y is not empty so we can define E_i without any assumption on i.

Theorem

Fix $x_0 \in X$. If, for every x_1 in a neighborhood of x_0, $f(x)$ tends to a limit in Y when x tends to x_1 while remaining in A then $E_i f$ is continuous at x_0.

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Theorem

Fix $x_0 \in X$. If, for every x_1 in a neighborhood of x_0, $f(x)$ tends to a limit in Y when x tends to x_1 while remaining in A then $E_i f$ is continuous at x_0.
If in addition $x_0 = i(a)$, f is continuous at a and i pulls back the topology of X to the topology of A then $E_i(f)(i(a)) = f(a)$.

Side propaganda

The assumption in the first part of the previous theorem can be written as

$$\exists U \in \mathcal{N}_{x_0}, \forall x \in U, \exists y \in Y, \forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \\ \forall a \in A, i(a) \in V \Rightarrow f(a) \in W.$$

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$$\forall a \in A, i(a) \in V \Rightarrow f(a) \in W.$$

In mathlib, the assumption is written as

$$\{x \mid \exists y, f_* i^* \mathcal{N}_x \le \mathcal{N}_y\} \in \mathcal{N}_{x_0}.$$

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- Big projects are good. Next one?